Price Twenty-Five Cents.

VOL. I. NO. II.

THE

MATHEMATICAL MONTHLY.

NOVEMBER, 1858.

EDITED BY

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CAMBRIDGE:
PUBLISHED BY JOHN BARTLETT.
LONDON:
TRÜBNER AND CO.
1858.

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A single Copy,	$3.00 per annum.
Two Copies to one Address,	5.00 "
Five Copies " "	11.00 "
Ten Copies " "	20.00 "

Payable invariably in advance.

JOHN BARTLETT, *Publisher.*

Entered according to Act of Congress in the year 1858, by
J. D. RUNKLE,
In the Clerk's Office of the District Court of the District of Massachusetts.

CAMBRIDGE:
ALLEN AND FARNHAM, ELECTROTYPERS AND PRINTERS.

THE

MATHEMATICAL MONTHLY.

Vol. I...NOVEMBER, 1858....No. II.

PRIZE PROBLEMS FOR STUDENTS.

I.

WHAT conditions ought the sides and angles of a parallelogram to satisfy, in order that a square may be inscribed in it?

II.

Measure the sides a, b, c of a plane triangle, and suppose that α, β, γ are respectively the errors of the sides. Find the influence of these errors upon the angles A, B, C.

III.

Two triangles, ABC and abc, are similar when the angles A and a are equal, and the sides BC and bc, opposite the equal angles, are to each other as the perimeters of the triangles.

IV.

Show that the product of six entire consecutive numbers cannot be the square of a commensurable number.

V.

If, through any point, secants be drawn to a curve of the second order, and through the two points where each secant intersects the

curve, tangents be drawn meeting each other, the locus of all such points of meeting is a straight line.

The solutions of these problems must be received by the 10th of January, 1859.

Credit for the prize problems will always be given with their solutions.

FIRST LESSONS IN NUMBER.

BY REV. THOMAS HILL.

As the senses are developed before the power of abstract thought, so concrete arithmetic should precede the arithmetic of pure numbers. The little wood-cuts, given in the most elementary books on arithmetic, are not sufficient for the purpose of conveying just ideas of number. For beginners in arithmetic, a quart of corn or beans is an almost indispensable apparatus. The sliding beads upon a wire frame, frequently used, are rendered comparatively worthless by having twelve beads upon a rod, in threes of one color. They should have ten beads upon a rod, five of each color.

One of the earliest points to which the attention of the child should be directed, is the difference between prime and composite numbers. The natural series of numbers from one to twelve, will afford food for thought and materials for amusement to a young scholar for many months. Let him take, for example, six beans, and arrange them in two groups of three each, this will show him that $2 \times 3 = 6$. Each of the three beans, in one pile, may take a companion from the other, and thus we shall have $3 \times 2 = 6$. From this example, the child may commence the induction that the value of the product is independent of the order of the factors. Dividing the six again into two threes, we may again reduce it to three twos

by simply subtracting one from each of the threes and putting these two ones together. From this the child may begin his inductions in regard to permutations and combinations. If one is now added to the whole, making seven, none of the properties of a composite number remain, but some of the truths concerning permutation and combination are unaffected.

The young child must not be expected, from his arrangement of beans upon his desk, to *deduce* the properties of numbers, but only to *see* them. It is the sense, rather than the reason, which is to be first exercised.

SUBTRACTION BY MULTIPLICATION AND ADDITION.

BY PLINY EARLE CHASE.

THE following rule will doubtless be new and interesting to such readers of the Mathematical Monthly as are not familiar with Mann and Chase's Arithmetics, or Chase's Common School Arithmetic.

Write nine times the subtrahend under the minuend, and add each figure of the upper number to the figures in the same place, and all the inferior places of the lower number, carrying as in ordinary addition. Proceed in this manner, stopping at the figure that falls immediately under the left hand figure of the minuend, and the result will be the difference sought.

Example. Required the difference between 9047 and 23615.

23615
81423 = 9 × 9047
14568

5 and 3 are 8; 1 and 2 and 3 are 6; 6 and 4 and 2 and 3 are 15; 1 to carry and 3 and 1 and 4 and 2 and 3 are 14; 1 to carry and 2 and 8 and 1 and 4 and 2 and 3 are 21, but as the 1 falls under the left hand figure of the minuend, we stop there, and find the true remainder to be 14568.

Demonstration. If M represents the minuend, S the subtrahend,

and x the number of figures in the minuend, the result obtained by our rule is

$$M + (\overline{10^x - 1} - 10^x)\, S = M - S.$$

NOTE ON THE EXTRACTION OF THE CUBE ROOT OF NUMBERS.

BY THE EDITOR.

In the Nouvelles Annales de Mathématiques for January, 1858, we find the following method for extracting the cube root of numbers, which ought, on account of its easy application, to be generally used. The editor remarks in the April number, that the method had previously been given in a work entitled *Calcul pratiques*, in which it is claimed as new. The reader will find the same process, entitled a new method, in the American edition of Young's Algebra, published as long ago as 1832. It may also be found in some of our arithmetics; and many teachers, undoubtedly, already know and use it.

It is well known, that, in the usual rule, the chief difficulty consists in forming the trial divisors, and the labor increases in a rapid ratio with the number of figures in the root. The method proposed reduces this part of the process to the simple addition of numbers already used. If r_1, r_2, r_3, &c. denote the successive figures of the root, it is known that r_1 is found by simple inspection from the first, or left hand period of the given number, N, and that $3\,r_1^2$ is the trial divisor for finding r_2.

As soon as r_2 is found, we subtract from N, the cube

$$\begin{aligned} (r_1 + r_2)^3 &= r_1^3 + 3\,r_1^2 r_2 + 3\,r_1 r_2^2 + r_2^3 \\ &= r_1^3 + (3\,r_1^2 + 3\,r_1 r_2 + r_2^2)\, r_2, \end{aligned} \tag{1}$$

in which the quantity in the parenthesis is the complete divisor. The trial divisor for finding r_3, is

$$3(r_1 + r_2)^2 = 3r_1^2 + 6r_1r_2 + 3r_2^2$$
$$(2) \qquad = (3r_1^2 + 3r_1r_2 + r_2^2) + (3r_1r_2 + r_2^2) + r_2^2.$$

Now, comparing (1) and (2), we see at once, that the successive trial and complete divisors are made from each other as follows: —

(3) 1st trial divisor $= 3r_1^2$.

(4) 1st complete divisor $= 3r_1^2 + (3r_1 + r_2)r_2$.

(5) 2nd trial divisor $= 3(r_1 + r_2)^2$
$$= 3r_1^2 + (3r_1 + r_2)r_2 + (3r_1 + r_2)r_2 + r_2^2.$$

This shows, that when we have once computed $(3r_1 + r_2)r_2$ to find (4) from (3), we use the same result to find (5) from (4), only adding r_2^2 besides. In this way any trial divisor is made from the preceding complete divisor, by the addition of results already found.

The following example will sufficiently illustrate the method, and show the facility with which it can be used.

Extract the cube root of 162467446993496.

$$\begin{array}{l|l}
162467446993496 & 54566 = \text{root} \\
125 & 3 \times 5^2 = 75 \cdot\cdot \\
\hline
37467 & \left\{ \begin{array}{r} 616 = (3 \times 5 + 4)4 \\ \hline 8116 \times 4 \\ \hline 16 \end{array} \right. \\
32464 & \\
\hline
5003446 & 3 \times 54^2 = 8748 \cdot\cdot \\
 & \left\{ \begin{array}{r} 8125 = (3 \times 54 + 5)5 \\ \hline 882925 \times 5 \\ \hline 25 \end{array} \right. \\
4414625 & \\
\hline
588821993 & 3 \times 545^2 = 891075 \cdot\cdot \\
 & \left\{ \begin{array}{r} 98136 = (3 \times 545 + 6)6 \\ \hline 89205636 \times 6 \\ \hline 36 \end{array} \right. \\
535233816 & \\
\hline
53588177496 & 3 \times 5456^2 = 89303808 \cdot\cdot \\
 & \left\{ \begin{array}{r} 982116 = (3 \times 5456 + 6)6 \\ \hline 8931362916 \times 6 \\ \hline 36 \end{array} \right. \\
53588177496 & \\
\hline
 & 3 \times 54566^2 = 8932345068
\end{array}$$

It will be observed, that in the parentheses, the numbers preceding the sign + are tens, and those after it units.

SIMPLIFICATION OF THE EXPRESSION $\sqrt{a \pm c\sqrt{d}}$.

BY JOHN M. RICHARDSON.

THE solution of equations of the fourth degree, involving only the even powers of the unknown quantity, gives rise to the expression $\sqrt{(a \pm c\sqrt{d})}$. It can often be reduced to a simpler form, and it is proposed here to investigate it, and to deduce from the investigation a simple rule independent of any formula for the reduction of all such expressions.

The square root of a binomial cannot be extracted so long as it retains its binomial form; but if it can be resolved into a trinomial, the terms being so related that twice the product of the square roots of the extremes shall be equal to the mean, the root of the expression is equal to the sum or difference of the square roots of the extremes.

Whenever the numerical values of a, c, and d are given, $a \pm c\sqrt{d}$, can easily be resolved into a trinomial; and even when the expression is purely algebraic, it can in many cases be simplified.

Now, $$a \pm c\sqrt{d} = (a - d) \pm c\sqrt{d} + d,$$

and if $(a - d)$ is a perfect square whose root is the half of c, there results

$$\sqrt{(a \pm c\sqrt{d})} = \tfrac{1}{2}c \pm \sqrt{d}.$$

Before enunciating the rule, some numerical and literal examples will be solved.

$$\sqrt{(55 \pm 14\sqrt{6})} = \sqrt{(49 \pm 14\sqrt{6} + 6)} = 7 \pm \sqrt{6}.$$

$$\sqrt{(29 - 12\sqrt{-7})} = \sqrt{[36 - 12\sqrt{-7} + (-7)]} = 6 - \sqrt{-7}.$$

$$\sqrt{(1 \pm 4\sqrt{-3})} = \sqrt{[4 \pm 4\sqrt{-3} + (-3)]} = 2 \pm \sqrt{-3}.$$

$$\sqrt{(c^2 + ab \pm 2c\sqrt{ab})} = \sqrt{(c^2 \pm 2c\sqrt{ab} + ab)} = c \pm \sqrt{ab}.$$

The rule, then, is simply this: *Subtract* d *from* a, *and if the remainder is a perfect square whose square root is one half of* c, *the root of the expression*

is $\frac{1}{2}c \pm d$. In other words, *When the expression can be simplified, the root is half the coefficient of the second radical plus or minus, the radical itself.* There are, occasionally, some apparent exceptions to the rule, when the expression can be really simplified by other processes. But the exceptions are only apparent, and result from the fact, that the expressions are not in proper form for the application of the rule. Sometimes a factor is under the second radical, which should be in the coefficient; sometimes one in the coefficient which should be under the radical. Thus,

$\sqrt{(6 \pm \sqrt{32})} = \sqrt{(-26 \pm \sqrt{32} + 32)}$, and the rule does not apply; but $\sqrt{32} = 4\sqrt{2}$, and substituting,

$$\sqrt{(6 \pm 4\sqrt{2})} = \sqrt{(4 \pm 4\sqrt{2} + 2)} = 2 \pm \sqrt{2}.$$

$\sqrt{(46 \pm 6\sqrt{5})} = \sqrt{(41 \pm 6\sqrt{5} + 5)}$; and the rule does not apply. But $6\sqrt{5} = 2\sqrt{45}$; substituting

$$\sqrt{(46 \pm 2\sqrt{45})} = \sqrt{(1 \pm 2\sqrt{45} + 45)} = 1 \pm \sqrt{45} = 1 \pm 3\sqrt{5}.$$

Of course there are many expressions which cannot be simplified; but when the expression can be reduced, the rule given is the shortest and the most direct for obtaining it in its simplest form.

NOTE ON DERIVATIVES.

BY REV. THOMAS HILL.

1. To find by the aid of derivatives the development of $(a+x)^n$. Let us suppose

$$(1) \qquad (a+x)^n = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$

the problem is to determine the values of A, B, C, &c.

Take the derivatives relatively to x, and also to a, denoting the latter by accenting the letters, that is, putting $A' = D_a A$, &c.

This gives

(2) $\quad n(a+x)^{n-1} = B + 2\,Cx + 3\,Dx^2 + 4\,Ex^3 + \&c.$

(3) $\quad n(a+x)^{n-1} = A' + B'x + C'x^2 + D'x^3 + \&c.$

As these are to be true for all values of x we may put $x = 0$ by which (1) gives us

$$a^n = A,$$

while (2) and (3) give

$$na^{n-1} = B = A'.$$

As the first members of (2) and (3) are equal, the second members are also, and will remain so, after subtracting the equal quantities B and A'. This gives

$$2\,Cx + 3\,Dx^2 + \&c. = B'x + C'x^2 + \&c.$$

Dividing by x we have

$$2\,C + 3\,Dx + \&c. = B' + C'x + \&c.$$

This by making $x = 0$ gives

$$2\,C = B';\ C = \tfrac{1}{2}B'.$$

A repetition of the process will give us the series,

$$A = a^n,$$
$$B = A' = na^{n-1},$$
$$C = \tfrac{1}{2}B' = \tfrac{1}{2}n(n-1)a^{n-2},$$
$$D = \tfrac{1}{3}C' = \tfrac{1}{3}.\tfrac{1}{2}n(n-1)(n-2)a^{n-3},$$
$$E = \tfrac{1}{4}D' = \tfrac{1}{4}.\tfrac{1}{3}.\tfrac{1}{2}n(n-1)(n-2)(n-3)a^{n-4},$$

and so on.

These values of A, B, C, &c., being substituted in (1) give us

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \&c.$$

2. To find, in derivatives of a function of x, the value of the function after x has been increased by a definite amount.

Let us denote any function of x by the expression $f(x)$, and the same function after adding h to x by the expression $f(x+h)$.

Let h be divided into an infinite number of equal parts, either of which $= i$. Then it is manifest, since

$$\frac{f(x+i)-f(x)}{i} = Df(x),$$

that

$$(4) \qquad f(x+i)=f(x)+iDf(x).$$

And if in this, we put x' for x,

$$f(x'+i)=f(x')+iDf(x').$$

Then putting $x' = x+i$, we have by (4)

$$f(x+2i)=f(x)+iDf(x)+iDf(x)+i^2D^2f(x),$$
$$=f(x)+2iDf(x)+i^2D^2f(x).$$

In like manner

$$f(x+3i)=f(x)+3iDf(x)+3i^2D^2f(x)+i^3D^3f(x).$$

And in general

$$f(x+ni)=f(x)+\frac{n}{1}Df(x)+\frac{n(n-1)i^2}{1.2}D^2f(x)+\&c.$$

And if we make n infinite so that $n-1$, $n-2$, $n-3$, &c. are each equal to n, we have

$$f(x+ni)=f(x)+\frac{ni}{1}Df(x)+\frac{(ni)^2}{1.2}D^2f(x)+\frac{(ni)^3}{1.2.3}D^3f(x)+\&c.$$

Putting $ni=h$, that is, making n the number of parts in h, gives

$$(5) \quad f(x+h)=f(x)+\frac{hDf(x)}{1}+\frac{h^2D^2f(x)}{1.2}+\frac{h^3D^3f(x)}{1.2.3}+\&c.$$

Making $x=0$, gives us

$$(6) \quad f(h)=f(0)+\frac{hDf(0)}{1}+\frac{h^2D^2f(0)}{1.2}+\frac{h^3D^3f(0)}{1.2.3}+\&c.$$

3. Equation (5) is called Taylor's theorem and equation (6) MacLaurin's theorem. Their great practical value is apparent at a single glance, and their metaphysical value is no less deserving of notice, as showing how each point of a curve contains in itself the law of the whole line. No student of ordinary algebra need find any difficulty in mastering these beautiful theorems as I have now presented them.

ON THE RELATION BETWEEN THE STATES OF MINIMUM AND EQUILIBRIUM.

BY JOHN PATERSON.

[*Continued from page* 14.]

1. Three equal translatory forces applied at the points A, B, C (fig. 4), and acting either all towards or all from the point O through the intervention of the rigid lines $A\,O, B\,O, C\,O$, every pair of which makes an angle of 120° at O, will equilibrate on that point. For, on O describe a circle with any radius, and draw a line $L\,L'$ in any direction through the centre O: the algebraical sum of the cosines $O\,c' + O\,c'' + O\,c'''$ formed by the unit measures $O\,a$, $O\,b$, $O\,c$, of the given forces, with the line $L\,L'$, is zero; the sum of the components of two of the unit forces is equal in magnitude and opposed in direction to the component of the third force in the same direction, and the direction of each force divides equally the angle formed by the two others.

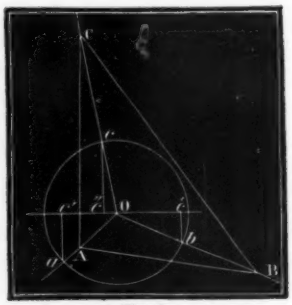

Fig 4.

The lines $O\,A$, $O\,B$, $O\,C$, may be prolonged indefinitely, and the points A, B, C taken anywhere on each: arcs to contain each an angle of 120°, described on $A\,B$, $B\,C$, $C\,A$ as chords, will always intersect in O, which is always, therefore, the point of aggregate minimum distances, so long as neither of the points passes over the point O.

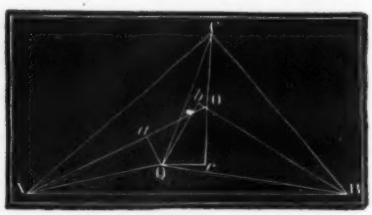

Fig. 5.

2. To show that O is the point of minimum distance, SIMPSON directs to assume some other point Q (fig. 5), and draw $Q\,a$, $Q\,b$, $Q\,c$ respectively perpendicular to $A\,O$, $B\,O$, $C\,O$; the sum of the prolongations $O\,b + O\,c$ is equal

to the curtailment Oa, and consequently

$$Aa+Bb+Cc=AO+BO+CO.$$

Draw the three hypothenuses AQ, BQ, CQ to form the right angled triangles $Aa\,Q$, $Bb\,Q$, $Cc\,Q$, and

$$AQ+BQ+CQ>AO+BO+CO.$$

3. The point of aggregate minimum distance having thus been found by geometry, the final solution of the problem appertains to ordinary trigonometry. Making $AO=x$, $BO=y$, $CO=z$, the triangles AOB, BOC, COA give the equations

$$c^2=x^2+y^2+xy,\ a^2=y^2+z^2+yz,\ b^2=z^2+x^2+zx,$$

because $\cos 120^\circ=-\frac{1}{2}$. The elimination being duly performed, the results are

$$x=\frac{c^2-b^2+a^2+\sqrt{[\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)]}}{(2(a^2+b^2+c^2)+6\sqrt{[\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)]})^{\frac{1}{2}}},$$

$$y=\frac{b^2-a^2+c^2+\sqrt{[\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(b^4+a^4+c^4)]}}{(2(a^2+b^2+c^2)+6\sqrt{[\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)]})^{\frac{1}{2}}},$$

$$z=\frac{a^2-c^2+b^2+\sqrt{[\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)]}}{(2(a^2+b^2+c^2)+6\sqrt{[\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)]})^{\frac{1}{2}}}.$$

This method of finding x, y, z is given by Mr. Perkins, in his Treatise on Trigonometry; and the elimination is exhibited at length in the Treatise on Algebra, by the same author.

Another method, but more circuitous and operose, is followed by some writers, which consists in equating the sum of the expressions for the areas of the included triangles AOB, BOC, COA, to the area of the including triangle ABC expressed in terms of the sides a, b, c, and then eliminating the unknown quantities by artifice as usual. The equation thus formed is

$$\frac{xy+yz+zx}{4}\sqrt{3}=(S(S-a)(S-b)(S-c))^{\frac{1}{2}}=M^2,$$

when calculated, because $\sin 120^\circ=\frac{\sqrt{3}}{2}$, and S is put for the semi-

perimeter of the triangle; and the results of the elimination are expressed by

$$x = \frac{1}{R}\left(\frac{b^2+c^2-a^2}{2} + \frac{2M^2}{\sqrt{3}}\right),$$

$$y = \frac{1}{R}\left(\frac{a^2+c^2-b^2}{2} + \frac{2M^2}{\sqrt{3}}\right),$$

$$z = \frac{1}{R}\left(\frac{a^2+b^2-c^2}{2} + \frac{2M^2}{\sqrt{3}}\right),$$

where R is put for

$$\sqrt{\left(\frac{a^2+b^2+c^2}{2} + 2M^2\sqrt{3}\right)}.$$

4. On the occasion of a railroad company in the western part of this State (New York) occupying three stations forming a triangle with sides measuring 60, 70 and 90 miles, and desiring to locate a depot of stores for the mutual accommodation of the stations at the point of minimum travel, this noted and somewhat difficult problem has recently been completely re-solved by a practical arithmetician, without calling in the aid of the higher geometry, and apparently entirely unaware that the question was extant in history. When it is brought to mind that the solution taxed the ingenuity of one of Italy's ablest geometers, it will reflect much credit upon the mathematical ability of the calculator who has thus achieved a new solution by the principles of arithmetic alone; while at the same time it adds another to the already numerous instances, so frequently happening in our country, of an almost useless expenditure of time and effort at the discovery of things that were brought to light centuries ago. An acquaintance with the labors of our predecessors is a useful introduction to the prosecution of new discovery; and mathematicians would do well sometimes to apply a phrase of SHAKSPEARE'S, "and look to the past, as well as to the future."

The minimum distances in the present example, calculated from

the formulæ of Mr. PERKINS, and agreeing to the sixth decimal with the values reported to the railroad company, are

$$x = 57.0841787$$
$$y = 46.6646085$$
$$z = 21.0164356$$
$$124.7652228 \text{ miles} = AO + BO + CO,$$

the aggregate minimum distance.

OVALS AND THREE-CENTRE ARCHES.

BY J. B. HENCK.

Application to certain Special Cases.

IN applying formula (1), page 26, we may make various suppositions, such as that the angle BED (Fig. 2) shall be 60°; that the first radius shall be a certain part of a; that a and b shall have a certain ratio to each other, &c. These suppositions will introduce additional equations which are to be combined with (1).

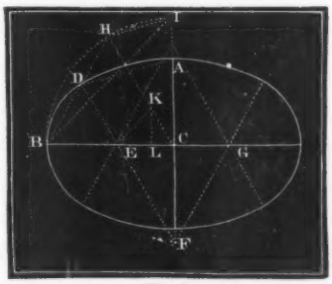

Fig 2.

Case I. When the angle BED (Fig. 2) is to be 60°. This gives the angles FEG and EGF each $= 60°$, and consequently $EF = 2EC$, or

$$r_1 - r = 2(a - r),$$
$$\therefore r_1 = 2a - r.$$

Substituting this value of r_1 in (1) we have

$$2ar + 4ab - 2br - 4ar + 2r^2 = a^2 + b^2,$$
$$\therefore r^2 - (a + b)r = \tfrac{1}{2}(a^2 + b^2 - 4ab),$$
$$\therefore r = \tfrac{1}{2}(a + b) \pm \tfrac{1}{2}(a - b)\sqrt{3},$$

$$\therefore\ r = a - \tfrac{1}{2}(a-b) - \tfrac{1}{2}(a-b)\sqrt{3},^{*}$$

$$\therefore\ r = a - \tfrac{1}{2}(a-b)(1+\sqrt{3}),$$

and $$r_1 = 2a - r = a + \tfrac{1}{2}(a-b)(1+\sqrt{3}).$$

The construction of this case is usually given as follows: With centre C and radius a draw the quadrant BHI. Make $BH = HC = a$. Join HI. Through A draw AD parallel to HI, meeting BH in D. Through D draw DEF parallel to HC, cutting BC in E, and AC produced in F. E and F are two of the centres.

Another construction, derived directly from the value of r just found, may be used. On HC make $KC = AI = a - b$. Through K draw KE parallel to BI. Through E draw DEF parallel to HC, cutting AC produced in F. E and F are two of the centres. By means of triangular rulers, having angles of 60° and 45°, this construction becomes remarkably simple. The correctness of this construction is evident. For if the perpendicular KL be dropped on EC, we have $LC = KC \cos 60° = \frac{1}{2}(a-b)$, and $EL = KL = KC \sin 60° = \frac{1}{2}(a-b)\sqrt{3}$ $\therefore\ EC = LC + EL = \frac{1}{2}(a-b)(1+\sqrt{3})$, and $BE = BC - EC = a - \frac{1}{2}(a-b)(1+\sqrt{3})$.

In a work in common use in this country, the centre E is obtained by laying off $EC = \frac{4}{3}(a-b)$. We have seen that it should be $\frac{1}{2}(1+\sqrt{3})(a-b) = 1.366\,(a-b)$ nearly. The error is too slight to be readily detected on a small figure on paper, but should an engineer lay out a large reservoir, for instance, by this rule (a misfortune that has actually happened), he would learn the value of knowing how to compute the radii accurately.

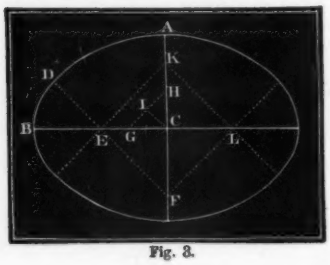

Fig. 3.

Case II. When the angle BED (Fig. 3) is to be 45°. Here the angles

* The lower sign alone of the double sign is to be used, since we know that r must be less than b, and, therefore, less than $\frac{1}{2}(a+b)$.

FEC and EFC are each $= 45°$, and consequently $CF = EC$ or

$$r_1 - b = a - r,$$
$$\therefore r_1 = a + b - r.$$

Substituting this value in (2), page 26, we have

$$a + b - r = a + \frac{(a-b)^2}{2(b-r)},$$
$$\therefore (b-r)^2 = \tfrac{1}{2}(a-b)^2,$$
$$\therefore b - r = \tfrac{1}{2}(a-b)\sqrt{2},$$
$$\therefore r = b - \tfrac{1}{2}(a-b)\sqrt{2},$$

and
$$\therefore r_1 = a + b - r = a + \tfrac{1}{2}(a-b)\sqrt{2}.$$

The construction is as follows: From C lay off CG and CH each equal to $a - b$. Join GH, and from C draw CI perpendicular to GH. Then $GI = \frac{1}{2}(a-b)\sqrt{2}$, and as $BG = b$, lay off $GE = GI$, and E will be one centre. The other centres F, K, and L are obtained by making CF, CK, and CL each equal to CE.

Case III. To draw an oval, so that the two radii may be as nearly equal as possible, that is, that $r_1 - r$ shall be a minimum.
By subtracting r from both sides of (2), we have

$$r_1 - r = a + \frac{(a-b)^2}{2b-2r} - r.$$

Differentiating, we have

$$D_r(r_1 - r) = \frac{2(a-b)^2}{4(b-r)^2} - 1 = 0,$$
$$\therefore (b-r)^2 = \tfrac{1}{2}(a-b)^2,$$
$$\therefore b - r = \tfrac{1}{2}(a-b)\sqrt{2},$$
$$\therefore r = b - \tfrac{1}{2}(a-b)\sqrt{2}.$$

This is the same value of r as that found in Case II.; so that to make the difference of the radii a minimum, the angle BED must be $45°$, and the construction is the same as that of Case II.

Case IV. When $a : b = 3 : 2$, and r is taken equal to $\frac{1}{2}a$. Here (2) gives

$$r_1 = a + \frac{(a - \frac{2}{3}a)^2}{\frac{1}{2}a - a} = a + \frac{(\frac{1}{3}a)^2}{\frac{1}{3}a} = \tfrac{4}{3}a = 2b.$$

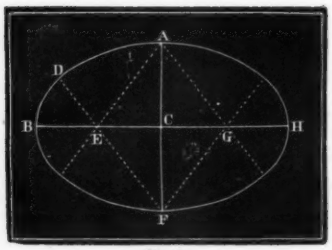

Fig. 4.

The construction of this case is extremely easy, and merits attention when the ratio of a to b can be assumed as above. The centres E and G (Fig. 4) bisect $B\,C$ and $C\,H$, and the other two centres A and F are at the extremities of the diameter $A\,F$.

Case V. When $D\,F$, the line passing through both centres (Fig. 5), is to be perpendicular to the whole chord $A\,B$. Here the triangle $E\,C\,F$ becomes similar to $A\,B\,C$, and we have

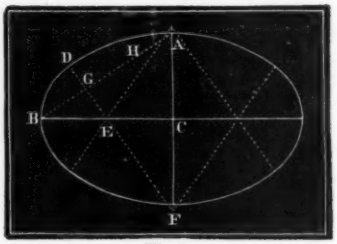

Fig. 5.

$$r_1 - b : a - r = a : b,$$

$$\therefore b r_1 - b^2 = a^2 - a r,$$

$$\therefore r_1 = \frac{a^2 + b^2 - a r}{b}.$$

If in this value of r_1 and in the first value given in (2) we put $a^2 + b^2 = c^2$, and then put the two values equal to each other, we have

$$\frac{c^2 - a r}{b} = \frac{c^2 - 2 a r}{2 b - 2 r},$$

$$\therefore 2 b c^2 - 2 a b r - 2 c^2 r + 2 a r^2 = b c^2 - 2 a b r,$$

$$\therefore r^2 - \frac{c^2 r}{a} = -\frac{b c^2}{2 a},$$

$$\therefore r = \frac{c^2}{2 a} \pm \sqrt{\frac{c^4 - 2 a b c^2}{4 a^2}},$$

$$\therefore r = \frac{c^2}{2 a} \pm \frac{c}{2 a}\sqrt{c^2 - 2 a b},$$

$$\therefore r = \frac{c^2}{2 a} \pm \frac{c}{2 a}(a - b),$$

$$\therefore r = \frac{c\,(c - a + b)}{2 a},$$

$$\therefore r_1 = \frac{c^2 - \frac{1}{2} c\,(c - a + b)}{b} = \frac{c\,(c + a - b)}{2 b}.$$

These values give rise to a simple construction often given. On

AB take $AH = a - b$, and bisect BH by the perpendicular DF, cutting BC in E, and AC, produced when necessary, in F. E and F are the centres. For the similar triangles EBG and ABC give

$$BE : BG = AB : BC,$$

or
$$r : \tfrac{1}{2}(c - a + b) = c : a,$$

$$\therefore r = \frac{c(c - a + b)}{2a};$$

and the similar triangles AFG and ABC give

$$AF : AG = AB : AC,$$

$$r_1 : \tfrac{1}{2}(c + a - b) = c : b,$$

$$\therefore r_1 = \frac{c(c + a - b)}{2b}.$$

ON THE DIVISIBILITY OF NUMBERS.

BY E. B. ELLIOTT.

ANY whole number, N, may take the usual decimal form

$$(1) \qquad \begin{aligned} N &= 10^p n_p + 10^{p-1} n_{p-1} + \ldots 10^2 n_2 + 10^1 n_1 + n_0, \\ &= \Sigma_p 10^p n_p, \end{aligned}$$

in which n_p, n_{p-1}, &c., are the digits in the order of their places, n_0 being the unit digit, n_1 the digit in the place of tens, and so on; $p+1$ denotes the number of digits, and Σ_p the sum taken relatively to p.

We wish a simple rule for determining by inspection when the number, N, is an exact multiple of a given integral divisor, D; and when not a multiple, for determining the number of units in the excess of N over the largest multiple of the divisor, which may be contained in it.

Any power p of 10 is equal to some multiple M of the divisor D, plus a remainder r_p; that is

$$(2) \qquad 10^p = M D + r_p,$$

in which M may be zero, or any integer, whatever, whether positive or negative. The numerical value of the remainder r_p may be essentially either positive or negative.

By substituting this value of 10^p in (1), it becomes

$$N = \Sigma_p (M\ D + r_p)\ n_p,$$
$$= \Sigma_p n_p M\ D + \Sigma_p n_p r_p,$$
$$= \Sigma_p n_p M\ D + (n_p r_p + n_{p-1} r_{p-1} + \ldots n_2 r_2 + n_1 r_1 + n_0 r_0).$$

It is manifest that whenever the quantity in the parenthesis is a multiple of the divisor D, the number N will be also.

It will commonly be desirable to have the numerical values of the remainders r_p, r_{p-1}, r_{p-2}, &c., the smallest integers possible, whether positive or negative. A convenient general method, for finding the numerical values of these remainders, will be to divide unity, followed by an indefinite number of zeros, by the given divisor D. The successive remainders will be the ones sought.

Example. Required, the numerical values of the remainders, r_p, r_{p-1}, r_{p-2}, &c., peculiar to the divisor seven.

$$\begin{array}{l|l} 7 & 1\,0\,0\,0\,0\,0\,0\,0\,0\,\cdots \\ \hline & 0\,1\,4\,3\,\bar{1}\,\bar{4}\,\bar{3}\,1\,4\,\cdots \quad \text{Quotient.} \\ & 1\,3\,2\,\bar{1}\,\bar{3}\,\bar{2}\,1\,3\,2\,\cdots \quad \text{Remainder.} \\ & r_0\,r_1\,r_2\,r_3\,r_4\,r_5\,r_6\,r_7\,r_8 \end{array}$$

Negative quotients and remainders have the sign minus written over them. In the usual form of division the quotients and remainders will be positive.

$$\begin{array}{l|l} 7 & 1\,0\,0\,0\,0\,0\,0\,0\,0\,\cdots \\ \hline & 0\,1\,4\,2\,8\,5\,7\,1\,4\,\cdots \quad \text{Quotient.} \\ & 1\,3\,2\,6\,4\,5\,1\,3\,2\,\cdots \quad \text{Remainder.} \\ & r_0\,r_1\,r_2\,r_3\,r_4\,r_5\,r_6\,r_7\,r_8 \end{array}$$

The number 454526 is not a multiple of seven, for the sum of once the unit digit (1×6), three times the next ($3 \times 2 = 6$), twice the next ($2 \times 5 = 10$), minus once the next ($-1 \times 4 = -4$), minus three times the next ($-3 \times 5 = -15$), minus twice

the next ($-2 \times 4 = -8$), is minus five ($6 + 6 + 10 - 4 - 15 - 8 = -5 = -1 \times 7 + 2$), which is not a multiple of seven; but the excess of minus five over the multiple of seven next smaller, that is, minus once seven, is two. Hence the excess of the number 454526 over the greatest contained multiple of seven is two.

For the negative values of the remainders ($-1, -3, -2$) may be substituted the positive values (6, 4, 5) obtained by adding seven to each negative value. But the repetend, in this case, would be one of six distinct places, namely (1, 3, 2, 6, 4, 5), instead of the simpler, the more easily remembered, and the more readily applied one of but three different places, namely (1, 3, 2, — 1, — 3, — 2), the successive series being alternately positive and negative. The following table gives values of the remainders corresponding to a few of the practically more important divisors; and it may readily be continued for any other divisors whatever.

Divisors.	*Remainders.*									
D.	r_0,	r_1,	r_2,	r_3,	r_4,	r_5,	r_6,	r_7,	r_8,	....
3.	1,	1,	1,	1,	1,	1,	1,	1,	1,	....
7.	1,	3,	2,	−1,	−3,	−2,	1,	3,	2,	....
	or 1,	3,	2,	6,	4,	5,	1,	3,	2,	....
8.	1,	2,	4,	0,	0,	0,	0,	0,	0,	....
9.	1,	1,	1,	1,	1,	1,	1,	1,	1,	....
11.	1,	−1,	1,	−1,	1,	−1,	1,	−1,	1,	....
	or 1,	10,	1,	10,	1,	10,	1,	10,	1,	....
13.	1,	−3,	−4,	−1,	3,	4,	1,	−3,	−4,	....
	or 1,	10,	9,	12,	3,	4,	1,	10,	9,	....
17.	1,	−7,	−2,	−3,	4,	6,	−8,	5,	−1,	....
	or 1,	10,	15,	14,	4,	6,	9,	5,	16,	....
99.	1,	10,	1,	10,	1,	10,	1,	10,	1,	....
101.	1,	10,	−1,	−10,	1,	10,	−1,	−10,	1,	....
	or 1,	10,	100,	91,	1,	10,	100,	91,	1,	....

The remainders peculiar to the number *seventeen* may take the form either of a positive repetend of sixteen places, or of repetends alter-

nately positive and negative of but eight places each, the complete series being

1, 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3, 13, 11, 8, 12, ...
or 1, —7, —2, —3, 4, 6, —8, 5, —1, 7, 2, 3, —4, —6, 8, —5, ...
or 1, 10, —(2, 3, —4) + 3 (2, 3, —4), —1, —10, + (2, 3, —4,) —3 (2, 3, —4), ...

The above general method for finding the numerical values of the remainders $(r_0, r_1, r_2, r_3, \ldots.)$ peculiar to any given divisor, accompanied by their numerical values for all integral divisors between 1 and 30, was given by the present writer in the year 1846 to classes of pupils at Lyons, in the State of New York.

In the fourth volume of the "Mémoires de la Société Impériale des Sciences Naturelles de Cherbourg," published in 1856, appears a paper entitled "Caractères de Divisibilité des Nombres Entires, par M. de Lapparent," in which it is stated that "the celebrated shepherd computer of Touraine, Henri Mondeux, in his last passage to Cherbourg in February, 1856, had offered for sale a pamphlet in which he indicated the characters of divisibility of numbers by the divisors comprised between 1 and 50. These very ingenious formulas were not accompanied by any theory."

"In fact, the former instructor of Mondeux, M. Jacobi, in the preface to the pamphlet in question, says, 'that his pupil had no knowledge of the principles upon which these formulas could be based, and that he himself avowed, in all humility, his ignorance of them.'" The object of the memoir of M. de Lapparent was to investigate these principles.

The properties of numbers which were indicated by Mondeux, so far as can be ascertained from the few examples given in the above-mentioned memoir, do not essentially vary from the theory and processes which we have suggested above, but differ widely from the very ingenious formulas of Lapparent.

Hereafter we purpose discussing the subject from another point of view, and one which will embrace the theory of M. de Lapparent.

QUERY BY MAJOR J. G. BARNARD.

Is there any *general* proof that

(1) $$\varphi(y + x\sqrt{-1}) + \varphi(y - x\sqrt{-1}),$$

and

(2) $$\frac{1}{\sqrt{-1}}\{\varphi(y + x\sqrt{-1}) - \varphi(y + x\sqrt{-1})\}$$

are in all cases *real quantities*, φ being any arbitrary function which is supposed to contain nothing imaginary in its form? Mr. Airy assumes it.

Answer by the Editor. Put $h = x\sqrt{-1}$, and develop $\varphi(y + h)$ and $\varphi(y - h)$ by Taylor's theorem. We get

(3) $$\varphi(y + h) = \varphi y + \varphi' y \cdot \frac{h}{1} + \varphi'' y \cdot \frac{h^2}{1.2} + \varphi''' y \cdot \frac{h^3}{1.2.3} + \&c.$$

(4) $$\varphi(y - h) = \varphi y - \varphi' y \cdot \frac{h}{1} + \varphi'' y \cdot \frac{h}{1.2} - \varphi''' y \cdot \frac{h^3}{1.2.3} + \&c.$$

Hence, by addition,

$$\varphi(y + h) + \varphi(y - h) = 2\varphi y + 2\varphi'' y \cdot \frac{h^2}{1.2} + 2\varphi'''' y \cdot \frac{h^4}{1.2.3.4} + \&c.$$

which contains only even powers of h, and is therefore always real, so long as φy is real. If instead of adding, we take the difference of these developments, the even powers of h disappear, and the result,

$$\varphi(y + h) - \varphi(y - h) = 2\varphi' y \cdot \frac{h}{1} + 2\varphi''' y \cdot \frac{h^3}{1.2.3} + 2\varphi^{v} y \cdot \frac{h^5}{1.2.3.4.5} + \&c.,$$

contains only odd powers; and therefore every term contains $\sqrt{-1}$

as a factor which may be divided out, so that (2) is always real for real values of φy.

If we denote the real parts of (3) and (4) by P, and the imaginary parts by $Q\sqrt{-1}$, they become

(5) $\qquad \varphi(y+h) = P + Q\sqrt{-1},$

(6) $\qquad \varphi(y-h) = P - Q\sqrt{-1}.$

Hence, if $y + x\sqrt{-1}$, and $y - x\sqrt{-1}$ are imaginary conjugates, like functions of these quantities are also.

The more general expression

$$\psi(u+v\sqrt{-1})\,\varphi(y+x\sqrt{-1}) \pm \psi(u-v\sqrt{-1})\,\varphi(y-x\sqrt{-1})$$

may, by (5) and (6), be written in the form

$$(P+Q\sqrt{-1})(P'+Q'\sqrt{-1}) \pm (P-Q\sqrt{-1})(P'-Q'\sqrt{-1}),$$

which is readily seen to be real or imaginary according as we use the sign plus or minus. Of course, it has been assumed in what precedes, that real functions of imaginary quantities can be developed by the same methods, and into the same forms, as real quantities; also, that reasoning based upon possible development is conclusive.

THE PHILOSOPHY OF ALGEBRAIC SIGNS.

BY PROF. W. D. HENKLE.*

ONE of the most important things for the mathematical student to understand is the true nature of the signs + and —. I propose to explain their use solely upon the doctrine that they are signs of operations whose effects are directly opposite. When + denotes addition, — denotes subtraction; when + denotes multiplication, — denotes division; when + denotes subtraction, — denotes addition. It must be remembered that a different idea is attached to *a*

* Editor of the Indiana School Journal, Indianapolis, Indiana.

and $+ a$, the former merely indicating a quantity, and the latter a specific operation. To subtract b from a is merely a simple arithmetical operation, and the operation may be indicated by $a - b$, but to subtract $+ b$ from a involves an additional idea, for it means the subtraction from a of the specific operation indicated by $+ b$. Let us assume that the quantity a is the result obtained by adding b to R, that is, $a = R + b$; now if the operation $+ b$ in $R + b$ is subtracted, omitted, or neglected, we have merely the quantity R. It is evident then, that R is the result obtained by subtracting the operation $+ b$ from a, that is, $R = a - (+ b)$. We can change $a - (+ b)$ into an arithmetical equivalent in the following manner: subtract b (not $+ b$) from a, and to the result add b (not $+ b$), and we have $a - b + b$, an expression which, considered in reference to the result obtained by performing the operations indicated, is equivalent to a; hence $a - (+ b)$ is equivalent to the result obtained by omitting in $a - b + b$ the operation $+ b$. This establishes the fact that $a - (+ b) = a - b$, or that $+ b$ subtracted from a gives the same result as b subtracted from a. If we desire to subtract $- b$ from a, we write $a + b - b$ for a, and then omit the operation $- b$, showing that the result obtained by subtracting $- b$ from a is the same as when b is added to a.

Let us now subtract $b - c$ from a. Considering b as a quantity, not as an operation, we have $a - b$ for the subtraction of b from a, and the operation $- c$ subtracted from $a - b + c - c$, or $a - b$, gives $a - b + c$.

Let us subtract 5 from 3. It is evident that this is impossible, for the less does not contain the greater; but 3 can be subtracted from 5, thus leaving 2 yet demanding the operation of subtraction. This fact is represented by $- 2$, that is, $3 - 5 = - 2$.

In Analytical Geometry, we have positive and negative abscissas, and ordinates. For instance, $x = + 3$, and $y = - 3$, means the

point reached by going on the axis of abscissas 3 to the right of the origin, and then going 3 below the axis of abscissas on a line parallel to the axis of ordinates. Of course it is purely conventional that + indicates motion to the right, or above, and — motion to the left, or below.

A *positive* quotient in division indicates the number of times that the divisor must be *subtracted* from the dividend to obtain zero; and a *negative* quotient, the number of times that the divisor must be *added* to the dividend to obtain zero.

In the case of exponents we have + indicating multiplication and — division. We remark, first, that a denotes a quantity, but a^1 denotes the multiplication of 1 by a; also a^2 denotes the multiplication of 1 by a twice; that is, is the product of 1, a, and a. If we now agree to extend this simple arithmetical idea by saying that a^{+3} shall be the same as a^3, we have the means of introducing negative exponents. Thus if a^{+3} denotes 1 *multiplied* by a three times, a^{-3} might consistently denote 1 divided by a three times. If the word *functionize* meant *multiply* or *divide*, we might define the exponent of a quantity to represent the number of times unity is to be functionized by that quantity. We see now how a^0 equals 1, for it means that *one* is to be functionized no time by a, and therefore it must still be 1.

It might be asked why it would not be philosophical to have a positive exponent denote involution, and a negative one, evolution; that is, why should not a^{-3} denote the cube root of a, if a^{+3} denotes the third power of a? I answer that such a representation would prevent the use of fractional exponents, and limit the rule in reference to the addition and subtraction of exponents in multiplication and division.

NOTE. — Perhaps it may not be considered proper to speak of subtracting $+b$ from a, unless a is changed to $+a$. This change is evidently legitimate, whenever necessary.

EXTENSION OF THE PRISMOIDAL FORMULA.

BY CHAUNCEY WRIGHT.

We have shown that the Prismoidal Formula is an exact definite integral for algebraic expressions of positive integer powers not exceeding the third degree, and that it is sufficiently accurate in all cases where the fourth order of differences may be neglected. We propose now to determine similar formulas, which shall include higher orders of differences.

If we represent by

$$f(x),\ f(x+h_1),\ f(x-h_1),\ f(x+h_2),\ f(x-h_2),\ \&c.$$

a series of sections made perpendicular to the axis of X, through a solid or plane figure, and at the distances $x, x+h_1, x-h_1$, &c. from the origin of x, then the expression

$$(A)\quad \begin{aligned} & h_n\,[a_n f(x-h_n)+\ldots+a_2 f(x-h_2)+a_1 f(x-h_1) \\ & +af x+a_1 f(x+h_1)+a_2 f(x+h_2)+\ldots+a_n f(x+h_n)], \end{aligned}$$

which contains $2n+1$ terms, $n+1$ coefficients a, a_1, a_2, &c., and n values h_1, h_2, &c., may be made to represent the contents of any portion of the solid or plane figure for which $f(x)$ is an algebraic expression of positive integer powers not exceeding the $(4n-1)$th degree.

Formula (A) may be written thus:—

$$h_n \Sigma_n a_n\,[f(x+h_n)+f(x-h_n)]+h_n\,a\,f(x),$$

and developed by Taylor's Theorem, it becomes

$$(B)\quad \left.\begin{aligned} & 2h_n\Sigma_n a_n\left[f(x)+\tfrac{h_n^2}{1.2}f^{\text{II}}(x)+\tfrac{h_n^4}{2.3.4}f^{\text{IV}}(x)+\&c.\right] \\ & \qquad +h_n\,a\,f(x) \end{aligned}\right\} = \left\{\begin{aligned} & 2h_n\Big[(\tfrac{1}{2}a+a_1+a_2+\&c.)f(x)+\tfrac{1}{2}(a_1h_1^2+a_2h_2^2+\&c.)f^{\text{II}}(x) \\ & \qquad +\frac{1}{2.3.4}(a_1h_1^4+a_2h_2^4+\&c.)f^{\text{IV}}(x)\Big]. \end{aligned}\right.$$

Now, the definite integral

$$\int \left[f(x+h_n) - f(x-h_n) \right] dx$$

becomes, when developed,

$$(C) \quad \begin{aligned} &2 \int \left[h_n f^{\text{I}}(x) + \frac{h_n^3}{2.3} \cdot f^{\text{III}}(x) + \frac{h_n^5}{2.3.4.5} f^{\text{V}}(x) + \&\text{c.} \right] dx = \\ &2 h_n \left[f(x) + \frac{h_n^2}{2.3} f^{\text{II}}(x) + \frac{h_n^4}{2.3.4.5} f^{\text{IV}}(x) + \&\text{c.} \right], \end{aligned}$$

and, if we make the corresponding terms of (B) and (C) equal, we have the following equations

$$\begin{aligned} 1 &= \tfrac{1}{2} a + a_1 + a_2 + \ldots + a_n \quad = \Sigma_n (a_n) + \tfrac{1}{2} a, \\ \frac{h_n^2}{3} &= \quad a_1 h_1^2 + a_2 h_2^2 + \ldots + a_n h_n^2 = \Sigma_n (a_n h_n^2), \\ \frac{h_n^4}{5} &= \quad a_1 h_1^4 + a_2 h_2^4 + \ldots + a_n h_n^4 = \Sigma_n (a_n h_n^4), \\ \frac{h_n^6}{7} &= \quad a_1 h_1^6 + a_2 h_2^6 + \ldots + a_n h_n^6 = \Sigma_n (a_n h_n^6), \end{aligned}$$

and so on.

By putting $r_1 = \frac{h_1}{h_n}$, $r_2 = \frac{h_2}{h_n}$, &c., these equations become

$$(D) \quad \begin{aligned} 1 &= \tfrac{1}{2} a + a_1 \quad + a_2 \quad + \ldots + a_n, \\ \tfrac{1}{3} &= \quad a_1 r_1^2 + a_2 r_2^2 + \ldots + a_n, \\ \tfrac{1}{5} &= \quad a_1 r_1^4 + a_2 r_2^4 + \ldots + a_n, \\ \tfrac{1}{7} &= \quad a_1 r_1^6 + a_2 r_2^6 + \ldots + a_n, \end{aligned}$$

and so on, in which a, a_1, a_2, &c., are the coefficients of the sections, and r_1, r_2, &c., the ratios of their distances from the middle section to the distance h_n of the extreme sections from the middle.

If by l we denote the whole distance between the extreme sections, then $h_n = \frac{l}{2}$.

The number of the equations (D) which can be satisfied is equal to the number of undetermined quantities $a, a_1, a_2, \ldots a_n$; $r_1, r_2, \ldots r_{n-1}$, that is, to $(n+1) + (n-1) = 2n$. But the $(2n+1)$th

terms of the expressions (B) and (C) contain the coefficients of the $4n$th derivative of $f(x)$, which are therefore the first not included in the equations (D); hence (A) may be made accurate for $(4n-1)$ derivatives of $f(x)$. When the sections are made at equal intervals, the distances h_1, h_2, &c., and the ratios r_1, r_2, &c., are determined by their number, and (A) can be made accurate only for as many derivatives of $f(x)$ as there are sections, that is, for $2n+1$. Since there are $2n$ equations (D) the last will be of the form $\frac{1}{4n-1} = a_1 r_1^{4n-2} + a_2 r_2^{4n-2} \ldots + a_n$, and the error of (A) for functions of two higher orders of derivatives, that is for functions which have $4n$, or $4n+1$ derivatives, is

$$\frac{2 h_n^{4n+1}}{1.2.3 \ldots 4n} \left(a_1 r_1^{4n} + a_2 r_2^{4n} + \ldots + a_n - \frac{1}{4n+1} \right) f^{[4n]} x.$$

EXAMPLES.

1. When there are only three sections, $n = 1$, and the equations (D) become $1 = \frac{1}{2} a + a_1$ and $\frac{1}{3} = a_1$, hence $a = \frac{4}{3}$ and (A) becomes $\frac{h_n}{3} \left[f(x-h_1) + 4 f x + f(x+h_1) \right] = \frac{l}{6} \left(B + 4 B'' + B''' \right)$, which is the common prismoidal formula.

2. When $n = 2$, there are five sections, and the equations (D) become

$$\begin{aligned} 1 &= \tfrac{1}{2} a + a_1 + a_2, \\ \tfrac{1}{3} &= a_1 r_1^2 + a_2, \\ \tfrac{1}{5} &= a_1 r_1^4 + a_2, \\ \tfrac{1}{7} &= a_1 r_1^6 + a_2. \end{aligned}$$

If these sections are made at equal intervals, then r_1 is determined, and only three of these equations can be satisfied; that is, only five orders of derivatives can be included by them. As the second and fourth sections will in this case bisect the intervals between the middle and the extreme sections, $r_1 = \frac{1}{2}$ and the equations

$$1 = \tfrac{1}{2} a + a_1 + a_2, \quad \tfrac{1}{3} = a_1 \tfrac{1}{4} + a_2, \quad \tfrac{1}{5} = a_1 \tfrac{1}{16} + a_2,$$

give $a = \frac{12}{45}$, $a_1 = \frac{32}{45}$, $a_2 = \frac{7}{45}$,

so that the formula (A) becomes

$$(1) \qquad \frac{l}{90}(7B + 32B' + 12B'' + 32B''' + 7B^{\text{IV}}),$$

and its error for functions of six or seven derivatives is

$$\frac{f^{\text{VI}}(x) \cdot l^7}{1935360}.$$

If r_1 be taken so that the coefficient a of the middle section shall disappear, then

$$r_1 = \sqrt{\tfrac{1}{3}}, \quad a_1 = \tfrac{5}{6}, \quad a_2 = \tfrac{1}{6},$$

and we obtain a formula of four sections,

$$(2) \qquad \frac{l}{12}(B + 5B' + 5B'' + B'''),$$

in which the interval between B' and B'' is $\frac{l}{\sqrt{5}}$.

This formula is accurate for five derivatives, and its error for functions of six or seven derivatives is

$$\frac{f^{\text{VI}}(x) \cdot l^7}{1512000}.$$

If r_1 is left indeterminate, then the solution of the four equations above will give

$$a = \tfrac{64}{90}, \quad a_1 = \tfrac{49}{90}, \quad a_2 = \tfrac{9}{90}, \quad r_1 = \sqrt{\tfrac{3}{7}},$$

and we obtain the formula of five sections

$$(3) \qquad \frac{l}{180}\left(9B + 49B' + 64B'' + 49B''' + 9B^{\text{IV}}\right),$$

for which the sections B' and B''' are at the distance $\frac{l}{2}\sqrt{\frac{3}{7}}$ from the middle section B''. This formula is accurate for seven derivatives, and its error for functions of eight or nine derivatives is

$$\frac{f^{\text{VIII}}(x) \cdot l^9}{237081600}.$$

This error and those of formulas (1) and (2) may be estimated in terms of the finite differences of $f(x)$ by the following transformations. If the length l be divided by sections into so many parts m that the corresponding differences of $f(x)$ of a higher order than the ith, may be neglected, then

$$\Delta^i f(x) = \left(\frac{l}{m}\right)^i f^i(x), \text{ or } l m^i \Delta^i f(x) = l^{i+1} f^i(x),$$

so that, for instance, the error of formula (3) may be written

$$\frac{l m^8 \Delta^8 f(x)}{237081600}.$$

Hence, if the number of parts m into which l is divided be about the eighth root of 237081600, or about 11, and if the corresponding value of $\Delta^8 f(x)$ be inappreciable, then formula (3) is sufficiently accurate. In the same way the accuracy of formulas (1) and (2) may be estimated in practice.

In the application of formulas (1), (2), and (3) to symmetrical figures, as for instance for determining the contents of casks, since the sections at equal distances from the middle section are equal, these formulas may be written thus: —

$$\frac{l}{45}(7B + 32B' + 6B''),$$

$$\frac{l}{6}(B + 5B'),$$

$$\frac{l}{90}(9B + 49B' + 32B'').$$

Cases in practice might arise in which sections at equal intervals could not be obtained. For such cases special formulas might be easily obtained from the equations (D).

In some future number of this journal we shall apply the formulas of the preceding discussion to a variety of problems in Engineering, Tonnage of Vessels, Cask Gauging, &c.

SOLUTION OF PROF. PEIRCE'S PROBLEM.

BY WILLIAM WATSON.

"WHAT curve is represented by the equation

$$\sin \varepsilon = a \operatorname{Cos} (\log r + b),$$

in which r denotes the radius vector, and ε the angle between r and the corresponding tangent?"

Since $\operatorname{Cos} \alpha = \frac{1}{2} (c^{\alpha} + c^{-\alpha})$, in which c denotes the Naperian base, we have

$$(1) \qquad \sin \varepsilon = \frac{a}{2} (c^{\log r + b} + c^{-\log r - b}) = \frac{a}{2} \left(c^{b} r + \frac{1}{c^{b} r} \right).$$

Let p denote the perpendicular dropped from the origin on the tangent; and since $p = r \sin \varepsilon$, (1) multiplied by r becomes

$$r \sin \varepsilon = p = \frac{a}{2} \left(c^{b} r^{2} + \frac{1}{c^{b}} \right). \quad \therefore D_r p = a c^{b} r.$$

But the radius of curvature, $\varrho = \frac{1}{2} D_p r^2 = r D_p r = \frac{1}{a c^{b}} = \text{constant}$. Therefore the curve is the circumference of a circle.

NOTE. — This form of $\varrho = \frac{1}{2} D_p r^2$ is not new, but may be deduced as follows. Let ds be an element of the curve, φ and τ the angles, made by r and the tangent, with the prime radius. Since $r \sin \varepsilon = p$, we find

$$\cos \varepsilon = \frac{\sqrt{r^2 - p^2}}{r}, \; d\varepsilon = \frac{d \sin \varepsilon}{\cos \varepsilon} = \frac{r\,dp - p\,dr}{r \sqrt{r^2 - p^2}}.$$

Also, $$ds = \frac{dr}{\cos \varepsilon} = \frac{r\,dr}{\sqrt{r^2 - p^2}}; \; d\varphi = \frac{ds \sin \varepsilon}{r} = \frac{p\,dr}{r \sqrt{r^2 - p^2}}.$$

But $\tau = \varphi + \varepsilon \; \therefore d\tau = d\varphi + d\varepsilon = \frac{dp}{\sqrt{r^2 - p^2}}$; and substituting these values of ds and $d\tau$ in the value of $\varrho = \frac{ds}{d\tau}$, we get

$$\varrho = \frac{r\,dr}{\sqrt{r^2 - p^2}} \cdot \frac{\sqrt{r^2 - p^2}}{dp} = \frac{r\,dr}{dp} = \tfrac{1}{2} D_p r^2.$$

Or thus. Let r_ϱ be the radius vector of the evolute corresponding to r of the curve; then from the geometry of the figure,

$r_\varrho^2 = (\varrho - p)^2 + r^2 - p^2 = \varrho^2 - 2p\varrho + r^2$. Differentiating $\varrho = \frac{1}{2} D_p r^2$.

ANOTHER SOLUTION.

BY CHAUNCEY WRIGHT.

The equation $$\sin \varepsilon = a \operatorname{Cos} (\log r + b), \qquad (1)$$
may be changed to the exponential form

$$\sin \varepsilon = \frac{a}{2} (c^{\log r} c^{b} + c^{-\log r} c^{-b}) = \frac{a c^{b} r}{2} + \frac{a}{2 c^{b} r};$$

and thence

$$r \sin \varepsilon = \frac{a c^{b} r^{2}}{2} + \frac{a}{2 c^{b}}. \qquad (2)$$

If we assume a polar axis, and suppose a normal to be drawn from any point of the curve to this axis, and denote by N the length of the normal, and by l the portion of the axis cut off by the normal; then in the triangle formed by r, l, and N the angle included by r and N is $\frac{1}{2}\pi - \varepsilon$, and by Trigonometry

$$l^{2} = N^{2} + r^{2} - 2 N r \cos (\tfrac{1}{2}\pi - \varepsilon) = N^{2} + r^{2} - 2 N r \sin \varepsilon;$$

so that

$$r \sin \varepsilon = \frac{r^{2}}{2 N} + \frac{N^{2} - l^{2}}{2 N}. \qquad (3)$$

Now, in this general equation such values of N and l may be substituted, as will satisfy equation (2) for any possible values of a and b. If we take

$$N_1 = \frac{1}{a c^{b}} \text{ and } \frac{N_1^{2} - l_1^{2}}{N_1} = \frac{a}{c^{b}} = a^{2} N_1,$$

and thence

$$l_1 = \frac{\sqrt{1 - a^{2}}}{a c^{b}},$$

and substitute these values in (3) we obtain (2); but these values are constant; hence the curve is a circle for all values of b, and for all values of a^2 less than unity, and since equation (1) is impossible when a is greater than unity, it is the equation of a circle for all possible values of its constants. The radius of the circle is $\frac{1}{a c^{b}}$, and the origin of r is at the distance $\frac{\sqrt{1 - a^{2}}}{a c^{b}}$ from the centre.

A THIRD SOLUTION.

BY GEORGE B. VOSE.

Put $\log r + b = u$; whence $D_u r = r$ (1). Let φ denote the angle which the radius vector makes with a given line, and we have the relation $D_r \varphi = \frac{\tan \varepsilon}{r}$. (2). Multiplying together (1) and (2) we obtain

$$D_u \varphi = \tan \varepsilon = \frac{a \operatorname{Cos} u}{\sqrt{1 - a^2 \operatorname{Cos}^2 u}},$$

the integral of which is

$$\sin (\varphi + \varphi_0) = \frac{a \operatorname{Sin} u}{\sqrt{1 - a^2}} = \frac{a}{2\sqrt{1 - a^2}} \left(c^b r - \frac{1}{c^b r} \right),$$

in which φ_0 is the constant. Passing from polar to rectangular coördinates by means of the formulas $x = r \cos (\varphi + \varphi_0)$, and $y = r \sin (\varphi + \varphi_0)$, we finally obtain

$$c^b y = \frac{a}{\sqrt{1 - a^2}} (c^{2b} (x^2 + y^2) - 1),$$

which is the equation of a circle.

PROBLEMS, BY PROF. PEIRCE.

Investigate the number of real roots in each of the following equations: —

I. $a^\theta + b^\theta + c = 0$,

II. $\sin^m \theta + a \sin^n \theta + b\theta + c = 0$,

III. $a^\theta + b\theta^n + c = 0$.

IV. Find the least sphere which shall contain a given system of points. This problem is an extension of the following: —

"It is required to find the least circle which shall contain a given system of points." — J. J. Sylvester, *The Quarterly Journal of Pure and Applied Mathematics.*

AN ACCOUNT OF THE COMET OF DONATI. 1858.

BY G. P. BOND.

THE following account of the Great Comet, discovered by DONATI on the 2d of June, 1858, has been prepared for the Mathematical Monthly, with the intention of furnishing correct information on a subject of universal interest, in language freed as far as possible from technical expressions. The popular character of the article renders a few preliminary words of explanation necessary in reference to the more distinctive phenomena presented in the motions and physical aspect of comets generally.

The first characteristic of these singular bodies is that of their being mainly, perhaps in most instances, entirely composed of an ill-defined gaseous or nebulous substance, endowed with properties so extraordinary, that it can scarcely be classed with matter, in the ordinary acceptation of the term. Of its extreme attenuation and lightness, there can be no question. The planets, and among them our earth, must again and again have traversed unharmed the tails of comets. In October last, the debris of the magnificent train of the comet which has just disappeared from our western skies, swept over the region occupied by the earth a few weeks earlier. Instances of more immediate proximity are of too common occurrence to allow us to suppose that we are always to escape an actual collision; but it is inconceivable that any disastrous consequences could ensue to our earth or its inhabitants, any more than from contact with sunlight or with the ether of the planetary spaces.

A second characteristic is that of internal condensation. All comets present this in a greater or less degree. Most of them have a minute stellar point, called the nucleus, which occupies the position of maximum density. There are others in which this latter feature is wholly wanting. But the number, in which it cannot be detected with a powerful telescope, is much smaller than has com-

monly been supposed. This centre of condensation, or brightest point is, with rare exceptions, placed on the side which is nearest to the sun. It is always, however, very close to the centre of gravity, as is proved by the fact of its motion about the sun, in accordance with the law of gravitation.

The nucleus itself is a minute point compared with the immense volume of light-giving substance, of which it is the controlling centre. Whether it is solid or not, is still undecided. As far as the eye alone is to be trusted, there are comets as truly solid as the planets or stars themselves. In size and weight, however, the true nuclei, apart from their surrounding nebulosity, are probably quite small, measured by the standard of the larger planets. Still it is possible that there may have been instances in which the mass of these bodies has been comparable with that of the earth, and yet they may have completed their circuit around the sun, leaving no appreciable trace of their disturbing influence — the only sure test by which their mass could be detected. The evidence, from the fact that the smaller stars shine freely even through the most condensed portions of comets, adduced by astronomical writers in proof of their transparency, and, by inference, of their extreme tenuity and lightness, has a certain value when applied to the class of feeble telescopic comets; but is scarcely applicable to one like that of the present year, which overpowered all but the brighter stars in the neighborhood of the nucleus by its superior brilliancy.

The feature next in importance to the nucleus is the train, or tail, as it is usually called (although often preceding the nucleus in its motion), projected at an immense distance from it, and usually, although by no means invariably, in a direction opposite to that of the sun. The agency of the nucleus in the formation of the train, but still more in the subsequent control which it retains over it, is one of the most curious phenomena presented in nature. Often, several of these appendages are seen radiating at once from the

same nucleus. The greatest variety in curvature of outline, length, brilliancy, and other peculiarities, is presented by different comets, or by the same one in different parts of its course. The portions near the axis are usually darker than the edges, giving at times the appearance of a division with a stream of light on either side.

The larger bodies of this class exhibit a wonderful complication of phenomena in the region contiguous to the nucleus. Of these, the most prominent are the interposition between the nucleus and the sun of one or more well-defined and rounded screens, or caps of dense nebulosity, called envelopes, partially but not entirely surrounding the nucleus, and the emission of streams or jets of luminosity, bright sectors, &c., in a direction inclined or opposite to that of the tail. With great variety of detail in other respects, these have all a well-marked tendency to appear in the first instance on the side of the nucleus next the sun. The great comet of the present year undoubtedly takes a foremost rank in respect of the multiplied and most curious changes which it has exhibited, and especially in the complete illustration which it has afforded of the origin, construction, and final dissipation of a succession of envelopes. In these phenomena, the process of the formation of the tail, from the substance in immediate contact with the nucleus, is intimately concerned. The astronomer, night by night, sees the work of evolution going on with an amazing rapidity, and seemingly in defiance of the best established properties of matter, the laws of gravitation and of inertia. The results are evident to all, but the secret cause is a profound mystery admirably calculated to stimulate speculation and intelligent investigation.

The following cut, Fig. 1,* with a brief recapitulation, will serve

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* The engraving has been copied from a drawing of the comet of DONATI, made by the writer, at the Observatory of Harvard College, October 2, 1858.

to illustrate the leading phenomena presented in a telescopic view of a large comet, *n* represents the star-like nucleus, *a* and *b* the outlines of two envelopes on the side of the nucleus towards the sun, *c* the diffused exterior nebulosity which is a never-failing attendant.

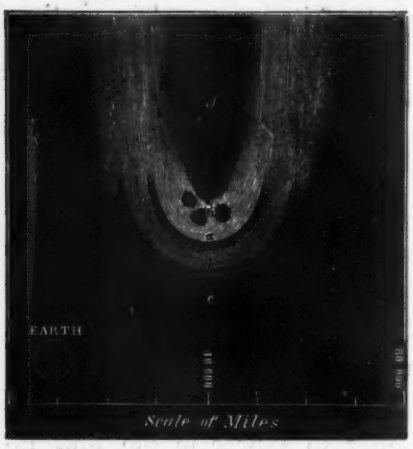

Fig. 1.

The dark axis, *d*, of the tail, is sometimes however absent, and again, at times, replaced by a bright ray. Ordinarily the convex and brightest side of the tail is presented to the region towards which the comet is moving.

As regards the motion of comets in space, it is a well-established fact, so far as our present means of observation extend, that their nuclei alone move in obedience to the attractive force of the sun and planets. This property, which has been recognized with consistency and uniformity, is not the least singular peculiarity of their constitution. Immense volumes of matter, apparently of the identical substance of the nucleus, go to compose the enveloping nebulosity and the tail, but from the moment of leaving the central body their motion is perfectly inexplicable without assuming them to be under the influence of laws of force which greatly modify that of gravitation.

The shape of the cometary orbits, described about the sun, is nearly that of a parabola, or of an elongated ellipse, with periods of revolution varying from a few years to many centuries. The point in the orbit which is nearest the sun is called the perihelion; the distance of this point from the sun, the perihelion distance, and the time of the comet's passing it, the perihelion passage.

Fig. 2 represents the form of a portion of the orbit of the great comet of the present year. The circles show the relative proportions of the orbits of Venus, the Earth, and Mars. The date, September 30, is the epoch of perihelion passage, the distance of the comet from the sun at that point being about fifty millions of miles.

Fig. 2.

For convenience of delineation the orbit is represented with its plane coincident with that of the earth's orbit. These planes have actually an inclination of sixty-three degrees to each other. The relative size and the form of the orbit, with the places of the comet in it, are given with sufficient exactness for the purpose of illustration. The arc described by the comet, during the period of its greatest brilliancy, is included between September 30 and October 10. It had, however, been detected by astronomers about three months earlier.

On the 2d of June, 1858, a faint nebulosity, slowly advancing towards the north, was descried by Donati at Florence, near the star λ *Leonis*. This was the earliest observation of the great comet of 1858. Its place at the time is shown in Fig. 2, its distance from the sun being then about two hundred millions of miles, while from the earth it was yet more remote. Being, at first, inclined to question whether it might not be identical with another comet just before seen in the same quarter of the heavens (the third comet of 1858), he communicated the intelligence of the discovery with a suitable reserve, as "perhaps new;" and in a second despatch he

said, "It is possible that this comet is the same as that discovered in America on the 2d of May." This conjecture, fortunately for DONATI, did not prove true; although the apprehension of the Italian astronomer, from the rival zeal of his transatlantic brethren, was not without reasonable foundation, for no sooner had the moon withdrawn from the evening sky so as to allow the comet to be seen, than it was detected almost simultaneously at three different points in America, each observer being at the time unaware of its previous discovery in Italy. It was seen by Mr. H. P. TUTTLE on the evening of the 28th of June, and an accurate determination of its place was made on the same night at the Observatory of Harvard College. On the 29th, it was detected by H. M. PARKHURST, Esq., of Perth Amboy, N. J., and on the 1st of July, by Miss MITCHELL, of Nantucket.

Three geocentric positions obtained on the 7th, 11th, and 13th of June, furnished DONATI with the means of computing approximate elements of the comet's motion, from which its interesting character was quickly recognized. Considerable difficulty was experienced, in fixing the precise time of perihelion passage, a most necessary condition in predicting its path as seen from the earth. While in other respects the results deduced by various computers were sufficiently accordant, they showed wide discrepancies in designating the place of the comet in the orbit. By the middle of August, however, its future course, and great increase of brightness in September and the early part of October, had been ascertained with entire certainty.

Up to this time it had remained a faint object, not even discernible by the unassisted eye. It was distinguished from ordinary telescopic comets only by the extreme slowness of its motion, in singular contrast with its subsequent career, and by the vivid light of the nucleus; the latter peculiarity was of itself prophetic of a splendid destiny.

Traces of a tail were noticed on the 20th of August, and on the 29th it was seen with the naked eye as a hazy star. For a few weeks it occupied a position in the heavens where it rose before the sun and set after it, becoming thus a conspicuous object both in the morning and evening sky. This circumstance gave rise to the erroneous notion that two different comets had appeared. The statement, which was widely circulated, that this was the return of the comet of 1264 and of 1556, supposed by some to be identical, is equally incorrect. If it has ever before been seen by man, it must have been far back in history, since the most recent computations assign a time of revolution of about twenty-five hundred years.

On the 6th of September was first noticed the curvature of the tail, which subsequently, at the time of its greatest expansion, became one of its most impressive features. It is remarkable that this peculiarity should have been strongly enough exhibited to be distinguished at the above date, when the earth was close to the plane of the comet's orbit. The observation cannot in fact be reconciled with the commonly received opinion that the curvature of the tail lies in the plane of motion about the sun.

The extraordinary changes exhibited in the neighborhood of the nucleus during the disruption of the envelopes, and by the train while the comet was in the part of its orbit nearest the sun, will be illustrated in the conclusion of this article by a series of woodcuts and of mezzotint plates. This method of description is the only one by which an adequate conception of the condition of the comet, during this critical period in its history, can be conveyed to the reader. Fortunately, a long succession of clear skies at Cambridge afforded uncommonly favorable opportunities for observation.

Editorial Items.

UNDER this head we shall include all such miscellaneous matter as we wish to communicate to the readers of the Monthly, and for which we do not find a more appropriate place. ..... PRIZE PROBLEMS FOR STUDENTS. — Since the issue of the first number of the Monthly, we have received several letters, and a complete set of solutions of the prize problems, from those who are not connected with any institution of learning, and yet wish to compete for the prizes. When we first decided to offer these prizes, we were desirous not to exclude any one justly entitled to compete for them, although he might not be so fortunate as to be connected with an institution of learning; but at that time it did not occur to us how we might extend them to all students, and still secure entire fairness among all the competitors. In the October number, we stated that the slips, containing the names of the competitors, and the institutions with which they are connected, must also be signed by their instructors, as evidence that the parties are fairly entitled to compete for the prizes. Now, if any student, not connected with an institution of learning, whether public or private, who wishes to compete for any of these prizes, will send the statement of a responsible person giving us the evidence that he is fairly entitled to be considered a competitor, we will withdraw the limitation..... It gives us pleasure to add the following names to our list of coöperators and contributors. *Major R. J. ADCOCK, Military Inst., Ky.; *Prof. WM. W. CLARKE, Genesee Wesleyan Seminary, Lima, N. Y.; J. N. LEWIS, Esq., Uniontown, Penn.; Rev. J. P. PERRY, Yarmouthport, Mass.; C. M. RUNK, Esq., Allentown, Penn.; EDWARD W. SERRELL, C. E., New York; P. H. THOMSON, C. E., Columbia, Tenn..... In the October number of the Monthly, on page 2, the right hand member of the first equation of Prize Problem I. should be -2 instead of $+2$. This will make the equation of the second degree as was intended. On page 19, lines -1 and -2, for ϱ_x read ϱ_x; page 19, lines -3 and -4, for ϱ_z read ϱ_y; page 21, line 10, for $1 + (D_x y)^2$ read $(1 + (D_x y)^2)^{\frac{1}{2}}$. These errata are corrected in the second edition..... Prof. PEIRCE's propositions on the distribution of points on a line have excited unusual interest. We have already received demonstrations from R. J. ADCOCK, W. P. G. BARTLETT, PLINY EARLE CHASE, E. B. ELLIOTT, SIMON NEWCOMB, and JAMES EDWARD OLIVER. They will receive early attention..... Contributions intended for any particular month should be received at least one month in advance..... A resolution, in favor of the Mathematical Monthly, moved by JAMES CRUIKSHANK, Esq., Editor of the New York Teacher, was passed by the New York State Teachers' Association at its late session, held at Lockport, N. Y. The certified copy ordered to be sent failed to reach us..... We wish to call the attention of Engineers to the Mathematical Monthly, and we trust that in the numbers already issued the evidence is not entirely wanting of its educational and practical value to their profession. See, for instance, the articles on Ovals and Three-Centre Arches, and on the Prismoidal Formula and its Extension..... We wish to call the attention of authors and publishers (especially of mathematical and scientific books) to the Mathematical Monthly as an advertising medium..... Books received. Elements of Mechanics. By J. B. CHERRIMAN, M. A., late Fellow of St. John's College, Cambridge, and Professor of Natural Philosophy in University College, Toronto. Maclear & Co., King Street East. 1858. Nouvelles Annales de Mathématiques for September, 1858. Paris: Mallet-Bachelier. Mensuration and Practical Geometry. By CHARLES H. HASWELL, Civil and Marine Engineer. New York: Harper & Brothers, Publishers, Franklin Square. 1858.

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